

# ON SEMI-OPEN CODES AND BI-CONTINUING ALMOST EVERYWHERE CODES

D. AHMADI DASTJERDI AND S. JANGJOOYE SHALDEHI

**ABSTRACT.** We will show that a system is synchronized if and only if it has a cover whose cover map is semi-open. Also, any factor code on an irreducible sofic shift is semi-open and the image of a synchronized system by a semi-open code is synchronized. On the other side, right-closing semi-open extension of an irreducible shift of finite type is of finite type. Moreover, we give conditions on finite-to-one factor codes to be open and show that any semi-open code on a synchronized system is bi-continuing a.e.. We give some sufficient conditions for a right-continuing a.e. factor code being right-continuing everywhere.

## 1. INTRODUCTION

A map  $\phi : X \rightarrow Y$  is called *open map* if images of open sets are open and is called *semi-open (quasi-interior)* if images of open sets have non-empty interior. In dynamical systems, they are of interest when they appear as factor map or as it is called factor code in symbolic dynamics. There are many classes of open and specially semi-open factor maps. For instance, any factor map between minimal compacta is semi-open [3] and all surjective cellular automata are semi-open [12, Theorem 2.9.3]. As one may expect, there are some strict restrictions for a factor map being open; nonetheless, some important classes do exist. As an example, if  $X$  and  $Y$  are compact minimal spaces, there are almost one-to-one minimal extensions  $X'$  and  $Y'$  by  $\psi_1$  and  $\psi_2$  respectively and an open factor map  $\phi' : X' \rightarrow Y'$  such that  $\psi_2 \circ \phi' = \phi \circ \psi_1$  [2]. Here, our investigation is mainly confined on semi-open factor codes between shift spaces. For open factor codes between shift spaces see [9, 10]. A stronger notion to a semi-open map is the notion of an irreducible map which will be of our interest as well. A map  $\phi : X \rightarrow Y$  is called *irreducible* if the only closed set  $A \subseteq X$  for which  $\phi(A) = \phi(X)$  is  $A = X$ .

A summary of our results is as follows. In section 3, we show that a coded system  $X$  is synchronized if and only if there is a cover  $\mathcal{G} = (G, \mathcal{L})$  so that its cover map  $\mathcal{L}_\infty$  is semi-open and if  $\mathcal{G} = (G, \mathcal{L})$  is an irreducible right-resolving cover for  $X$  with a magic word, then  $\mathcal{L}_\infty$  is semi-open (Theorem 3.7). Also, when  $\mathcal{G}$  is Fischer cover,  $\mathcal{L}_\infty$  is irreducible (Theorem 3.8). Furthermore, any factor code on an irreducible sofic shift is semi-open (Corollary 3.11). Theorem 4.2 implies that a right-closing semi-open extension of an irreducible shift of finite type is of finite type. Theorem 4.3 shows that the image of a synchronized system is again synchronized under a semi-open code and so providing a class of non-semi-open codes, mainly, codes factoring synchronized systems over non-synchronized ones.

---

2010 *Mathematics Subject Classification.* 37B10, 37B40, 05C50.

*Key words and phrases.* semi-open, almost one-to-one, shift of finite type, sofic, synchronized, continuing code.

We also consider bi-continuing and bi-continuing almost everywhere (a.e.) codes. A right-continuing or u-resolving code is a code which is surjective on each unstable set and it is a natural dual version of a right-closing code and plays a fundamental role in the class of infinite-to-one codes [7]. In Theorem 4.13, we give conditions on bi-continuing a.e. factor codes to be open and also by Corollary 4.14 and Theorem 4.15, we give some sufficient conditions for a right-continuing a.e. factor code being right-continuing everywhere. Finally, we show that any semi-open code on a synchronized system is bi-continuing a.e. (Theorem 4.16).

## 2. BACKGROUND AND NOTATIONS

Let  $\mathcal{A}$  be a non-empty finite set. The full  $\mathcal{A}$ -shift denoted by  $\mathcal{A}^{\mathbb{Z}}$ , is the collection of all bi-infinite sequences of symbols from  $\mathcal{A}$ . A block (or word) over  $\mathcal{A}$  is a finite sequence of symbols from  $\mathcal{A}$ . The *shift map* on  $\mathcal{A}^{\mathbb{Z}}$  is the map  $\sigma$  where  $\sigma(\{x_i\}) = \{y_i\}$  is defined by  $y_i = x_{i+1}$ . The pair  $(\mathcal{A}^{\mathbb{Z}}, \sigma)$  is called the *full shift* and any closed invariant set of that is called a *shift space* over  $\mathcal{A}$ .

Denote by  $\mathcal{B}_n(X)$  the set of all admissible  $n$ -words and let  $\mathcal{B}(X) = \bigcup_{n=0}^{\infty} \mathcal{B}_n(X)$  be the *language* of  $X$ . For  $u \in \mathcal{B}(X)$ , let the *cylinder*  $[u]$  be the set  $\{x \in X : x_{[l, l+|u|-1]} = u\}$ . For  $l \geq 0$  and  $|u| = 2l + 1$ ,  $[u]$  is called a *central*  $2l + 1$  cylinder.

Let  $\mathcal{A}$  and  $\mathcal{D}$  be alphabets and  $X$  a subshift over  $\mathcal{A}$ . For  $m, n \in \mathbb{Z}$  with  $-m \leq n$ , define the  $(m + n + 1)$ -*block map*  $\Phi : \mathcal{B}_{m+n+1}(X) \rightarrow \mathcal{D}$  by

$$(2.1) \quad y_i = \Phi(x_{i-m}x_{i-m+1}\dots x_{i+n}) = \Phi(x_{[i-m, i+n]})$$

where  $y_i$  is a symbol in  $\mathcal{D}$ . The map  $\phi = \Phi_{\infty}^{[-m, n]} : X \rightarrow \mathcal{D}^{\mathbb{Z}}$  defined by  $y = \phi(x)$  with  $y_i$  given by 2.1 is called the *code* induced by  $\Phi$ . If  $m = n = 0$ , then  $\phi$  is called *1-block code* and  $\phi = \Phi_{\infty}$ . An onto code  $\phi : X \rightarrow Y$  is called a *factor code*.

A point  $x$  in a shift space  $X$  is *doubly transitive* if every block in  $X$  appears in  $x$  infinitely often to the left and to the right. Let  $\phi : X \rightarrow Y$  be a factor code. If there is a positive integer  $d$  such that every doubly transitive point of  $Y$  has exactly  $d$  pre-images under  $\phi$ , then we call  $d$  the *degree* of  $\phi$ .

A code  $\phi : X \rightarrow Y$  is called *right-closing* (resp. *right-continuing*) if whenever  $x \in X$ ,  $y \in Y$  and  $\phi(x)$  is left asymptotic to  $y$ , then there exists at most (resp. at least) one  $\bar{x} \in X$  such that  $\bar{x}$  is left asymptotic to  $x$  and  $\phi(\bar{x}) = y$ . A *left-closing* (resp. *left-continuing*) code is defined similarly. If  $\phi$  is both left and right-closing (resp. continuing), it is called *bi-closing* (resp. *bi-continuing*). An integer  $n \in \mathbb{Z}^+$  is called a *(right-continuing) retract* of a right-continuing code  $\phi : X \rightarrow Y$  if, whenever  $x \in X$  and  $y \in Y$  with  $\phi(x)_{(-\infty, 0]} = y_{(-\infty, 0]}$ , we can find  $\bar{x} \in X$  such that  $\phi(\bar{x}) = y$  and  $x_{(-\infty, -n]} = \bar{x}_{(-\infty, -n]}$  [9].

Let  $G$  be a directed graph and  $\mathcal{V}$  (resp.  $\mathcal{E}$ ) the set of its vertices (resp. edges) which is supposed to be countable. An *edge shift*, denoted by  $X_G$ , is a shift space which consist of all bi-infinite sequences of edges from  $\mathcal{E}$ . A graph  $G$  is called *locally finite*, if it has finite out-degree and finite in-degree at any vertex. Recall that  $X_G$  is locally compact if and only if  $G$  is locally finite.

Let  $v \sim w$  be an equivalence relation on  $\mathcal{V}$  whenever there is a path from  $v$  to  $w$  and vice versa. For an equivalence class, consider all vertices together with all edges whose endpoints are in that equivalence class. Thus a subgraph called the (irreducible) *component* of  $G$  associated to that class arises; and if it is non-empty, then an irreducible subshift of  $X_G$ , called (irreducible) component of  $X_G$  is defined.

There is no subshift of  $X_G$  which is irreducible and contains a component of  $X_G$  properly.

A labeled graph  $\mathcal{G}$  is a pair  $(G, \mathcal{L})$  where  $G$  is a graph and  $\mathcal{L} : \mathcal{E} \rightarrow \mathcal{A}$  its labeling. Associated to  $\mathcal{G}$ , a space

$$X_{\mathcal{G}} = \text{closure}\{\mathcal{L}_{\infty}(\xi) : \xi \in X_G\} = \overline{\mathcal{L}_{\infty}(X_G)}$$

is defined and  $\mathcal{G}$  is called a *presentation* (or *cover*) of  $X_{\mathcal{G}}$ . When  $G$  is a finite graph and hence compact,  $X_{\mathcal{G}} = \mathcal{L}_{\infty}(X_G)$  is called a *sofic shift*. Call  $F(I) = \{u : u \text{ is the label of some paths starting at } I\}$  the *follower set* of  $I$ .

An irreducible sofic shift is called *almost-finite-type* (AFT) if it has a bi-closing presentation [11]. A (possibly reducible) sofic shift that has a bi-closing presentation is called an *almost Markov* shift. A shift space  $X$  has *specification with variable gap length* (SVGL) if there exists  $N \in \mathbb{N}$  such that for all  $u, v \in \mathcal{B}(X)$ , there exists  $w \in \mathcal{B}(X)$  with  $uwv \in \mathcal{B}(X)$  and  $|w| \leq N$ .

A word  $v \in \mathcal{B}(X)$  is *synchronizing* if whenever  $uv, vw \in \mathcal{B}(X)$ , we have  $uvw \in \mathcal{B}(X)$ . An irreducible shift space  $X$  is a *synchronized system* if it has a synchronizing word. Let  $\mathcal{G} = (G, \mathcal{L})$  be a labeled graph. A word  $w \in \mathcal{B}(X_{\mathcal{G}})$  is a *magic* word if all paths in  $G$  presenting  $w$  terminate at the same vertex.

A labeled graph  $\mathcal{G} = (G, \mathcal{L})$  is *right-resolving* if for each vertex  $I$  of  $G$  the edges starting at  $I$  carry different labels. A *minimal right-resolving presentation* of a sofic shift  $X$  is a right-resolving presentation having the fewest vertices among all right-resolving presentations of  $X$ . It is unique up to isomorphism [11, Theorem 3.3.18] and is called the *Fischer cover* of  $X$ .

Now we review the concept of the Fischer cover for a not necessarily sofic system developed in [8]. Let  $x \in \mathcal{B}(X)$  and call  $x_+ = (x_i)_{i \in \mathbb{Z}^+}$  (resp.  $x_- = (x_i)_{i < 0}$ ) the *right* (resp. *left*) *infinite X-ray*. For  $x_-$ , its follower set is defined as  $\omega_+(x_-) = \{x_+ \in X^+ : x_-x_+ \text{ is a point in } X\}$ . Consider the collection of all follower sets  $\omega_+(x_-)$  as the set of vertices of a graph  $X^+$ . There is an edge from  $I_1$  to  $I_2$  labeled  $a$  if and only if there is a  $X$ -ray  $x_-$  such that  $x_-a$  is a  $X$ -ray and  $I_1 = \omega_+(x_-)$ ,  $I_2 = \omega_+(x_-a)$ . This labeled graph is called the *Krieger graph* for  $X$ . If  $X$  is a synchronized system with synchronizing word  $\alpha$ , the irreducible component of the Krieger graph containing the vertex  $\omega_+(\alpha)$  is called the *(right) Fischer cover* of  $X$ .

The entropy of a shift space  $X$  is defined by  $h(X) = \lim_{n \rightarrow \infty} (1/n) \log |\mathcal{B}_n(X)|$ . A component  $X_0$  of a shift space  $X$  is called *maximal* if  $h(X_0) = h(X)$ .

### 3. OPEN, SEMI-OPEN AND IRREDUCIBLE CODES

We start with some necessary lemmas.

**Lemma 3.1.** [5, Lemma 1.4] *Assume  $\phi : X \rightarrow Y$  and  $\psi : Y \rightarrow Z$  are surjections such that  $\psi \circ \phi$  is semi-open. Then,  $\psi$  is semi-open as well. Moreover, if  $\psi$  is irreducible, then also  $\phi$  is semi-open.*

A continuous map  $\phi : X \rightarrow Y$  is *almost 1-to-1* if the set of the points  $x \in X$  such that  $\phi^{-1}(\phi(x)) = \{x\}$  is dense in  $X$ .

**Lemma 3.2.** *Let  $\phi : X \rightarrow Y$  and  $\psi : Y \rightarrow Z$  be two factor codes such that  $\psi \circ \phi$  is irreducible. Then,  $\phi$  and  $\psi$  are irreducible.*

*Proof.* If  $\phi$  or  $\psi$  is not irreducible, then it is not almost 1-to-1 [14, Theorem 10.2] and so  $\psi \circ \phi$  cannot be almost 1-to-1 either, a contradiction.  $\square$

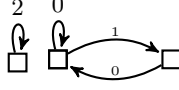
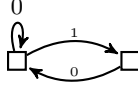
FIGURE 1. The cover of  $X$ .

FIGURE 2. The Fischer cover of the golden shift.

**Lemma 3.3.** [10, Lemma 1.2] *A code  $\phi : X \rightarrow Y$  between shift spaces is open if and only if for each  $l \in \mathbb{N}$ , there is  $k \in \mathbb{N}$  such that whenever  $x \in X$ ,  $y \in Y$  and  $\phi(x)_{[-k, k]} = y_{[-k, k]}$ , we can pick  $\bar{x} \in X$  with  $\bar{x}_{[-l, l]} = x_{[-l, l]}$  and  $\phi(\bar{x}) = y$ .*

So if  $\phi$  is open, then for any  $l \in \mathbb{N}$  we can find  $k \in \mathbb{N}$  such that the image of a central  $2l + 1$  cylinder in  $X$  consists of central  $(2k + 1)$  cylinders in  $Y$ . We say that  $\phi$  has a *uniform lifting length* if for each  $l \in \mathbb{N}$ , there exists  $k$  satisfying the above property such that  $\sup_l |k - l| < \infty$ . Let us formally state a definition for a semi-open code.

**Definition 3.4.** A code  $\phi : X \rightarrow Y$  between shift spaces is semi-open if for each  $l \in \mathbb{N}$ , there is  $k \in \mathbb{N}$  such that the image of a central  $2l + 1$  cylinder in  $X$  contains a central  $(2k + 1)$  cylinder in  $Y$ .

Note that the definition of uniform lifting length extends naturally to semi-open codes.

**Lemma 3.5.** *Let  $X$  and  $Y$  be shift spaces with  $Y$  irreducible. If  $\phi : X \rightarrow Y$  is a semi-open code, then it is onto and so a factor code.*

*Proof.* Since  $\phi$  is semi-open,  $\phi(X)$  contains a non-empty open set  $U \subseteq Y$ . Therefore, since  $(Y, \sigma)$  is topologically transitive,  $\bigcup_{n=-\infty}^{\infty} \sigma^n(U) = Y$  [13]. On the other hand,  $\phi(X)$  is a  $\sigma$ -invariant subset of  $Y$ . Thus  $\phi$  is onto.  $\square$

Our next results give situations where the factor code on covers are semi-open. In all of them, irreducibility is an important hypothesis. The following example shows that non-semi-openness may occur quite easily in reducible covers.

**Example 3.6.** Let  $X$  be a shift space presented by Figure 1 and  $Y$  the golden shift given by Figure 2. Define  $\Phi : \mathcal{B}_1(X) \rightarrow \{0, 1\}$  as,

$$\Phi(w) = \begin{cases} 1 & w = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and let  $\phi : X \rightarrow Y$  be the code induced by  $\Phi$ . Since  $[2] = 2^\infty$ ,  $\phi([2]) = 0^\infty$  and this shows that  $\phi$  is not semi-open. Note that  $\phi$  is a finite-to-one code.

**Theorem 3.7.** *A coded system  $X$  is synchronized if and only if there is a cover  $\mathcal{G} = (G, \mathcal{L})$  so that  $\mathcal{L}_\infty$  is semi-open. In fact, any irreducible right-resolving cover of a synchronized system with a magic word is semi-open.*

*Proof.* Let  $X$  be synchronized with a magic word  $\alpha$  and let  $\mathcal{G} = (G, \mathcal{L})$  be an irreducible right-resolving cover for  $X$ . Suppose  $[\pi] = [e_{-l} \cdots e_l] \in \mathcal{B}(X_G)$  is a central  $2l+1$  cylinder in  $X_G$  with  $\mathcal{L}(e_i) = a_i$ ,  $-l \leq i \leq l$ ; thus  $\mathcal{L}_\infty([\pi]) \subseteq [a_{-l} \cdots a_l]$ . Choose  $\lambda \in \mathcal{B}(X_G)$  such that  $\mathcal{L}(\lambda) = \alpha$ . Since  $X_G$  is irreducible, there are two paths  $\gamma$  and  $\xi$  in  $G$  such that  $\pi' = \lambda \xi \pi \gamma \lambda \in \mathcal{B}(X_G)$ . Now set  $w = \mathcal{L}(\pi') = \alpha \mathcal{L}(\xi) a_{-l} \cdots a_l \mathcal{L}(\gamma) \alpha$  and note that by the fact that  $\alpha$  is magic and  $\mathcal{G}$  is right-resolving, if  $w = \mathcal{L}(\pi'')$ , then  $\pi'' = \pi'$ . So  $[w] \subseteq \mathcal{L}_\infty([\pi'])$  and we are done.

For the converse let  $\mathcal{G} = (G, \mathcal{L})$  be a cover for  $X$  with  $\mathcal{L}_\infty$  semi-open. Since  $X_G$  is a Polish space,  $\mathcal{L}_\infty(X_G)$  is analytic and so there is an open set  $O \subseteq X$  and a first category set  $P \subseteq X$  so that  $\mathcal{L}_\infty(X_G) = O \Delta P$ . Therefore,

$$\mathcal{L}_\infty(X_G) = \bigcup_{n \in \mathbb{Z}} \sigma^n(O \Delta P) \supseteq \bigcup_{n \in \mathbb{Z}} \sigma^n(O) - \bigcup_{n \in \mathbb{Z}} \sigma^n(P).$$

Since  $\mathcal{L}_\infty$  is semi-open,  $O$  is non-empty. This means that the first union on right is a non-empty open dense set. As a result,  $\mathcal{L}_\infty(X_G)$  is residual and  $X$  synchronized [8, Theorem 1.1].  $\square$

When  $\mathcal{G}$  is the Fischer cover for a synchronized system, we will have a stronger conclusion.

**Theorem 3.8.** *If  $X$  is a synchronized system with the Fischer cover  $\mathcal{G} = (G, \mathcal{L})$ , then  $\mathcal{L}_\infty$  is irreducible.*

*Proof.* Let  $F$  be a proper closed subset of  $X_G$ . Then,  $F' = X_G \setminus F$  is open and there is a doubly transitive point  $x \in F'$ . But the degree of  $\mathcal{L}_\infty$  is one [11, Proposition 9.1.6] and so  $\mathcal{L}_\infty(x) \notin \mathcal{L}_\infty(F)$  which means that  $\mathcal{L}_\infty(F) \neq X$ .  $\square$

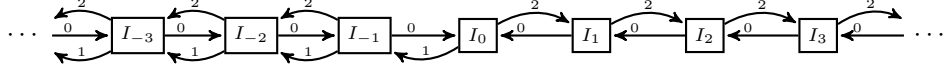
For sofics, when considering finite covers, a rather weaker hypothesis holds the same conclusion as Theorem 3.7.

**Theorem 3.9.** *Suppose  $X$  is sofic and  $\mathcal{G} = (G, \mathcal{L})$  an irreducible finite cover. Then,  $\mathcal{L}_\infty$  is semi-open.*

*Proof.* Suppose  $\mathcal{L}_\infty$  is not semi-open. So there is a cylinder  $[u_0]$  in  $X_G$  such that  $\mathcal{L}_\infty([u_0])$  does not have any central cylinder. Without loss of generality, assume that  $|u_0| = 1$ . Then, there must exist another edge  $u'_0 \in \mathcal{E}(G)$  such that  $\mathcal{L}(u'_0) = \mathcal{L}(u_0)$ ; otherwise,  $[\mathcal{L}(u_0)] \subseteq \mathcal{L}_\infty([u_0])$  and this contradicts our assumption. But to rule out  $[\mathcal{L}(u_0)] \subseteq \mathcal{L}_\infty([u_0])$  completely, one must have either  $F(\mathcal{L}(u_0)) \subsetneq F(\mathcal{L}(u'_0))$  or  $P(\mathcal{L}(u_0)) \subsetneq P(\mathcal{L}(u'_0))$ . Assume the former and suppose  $f \in F(u_0)$ . Then, there must be an edge  $f' \in F(u'_0)$  such that  $\mathcal{L}(f') = \mathcal{L}(f)$  and yet another path  $f'' \in F(u'_0)$  such that  $\mathcal{L}(f'') \notin F(I_0)$  where  $I_0 = t(u_0)$ .

Now choose any path  $u_1 = u_0 w_0 u'_0$ ,  $t(u_1) = t(u'_0) = I_1$  and observe that since we cannot have  $[\mathcal{L}(u_1)] \subseteq \mathcal{L}_\infty([u_0])$ , there is another path  $u'_1$  labeled as  $u_1$  whose last edge is  $u''_1$  with  $\mathcal{L}(u''_1) = \mathcal{L}(u'_0) = \mathcal{L}(u_0)$  such that  $I_2 = t(u''_1)$  has at least one path more than  $I_1$ . Construct  $I_n$  inductively. Then, the outer edges of  $I_n$  increases by  $n$  which is absurd for a finite graph such as  $G$ .  $\square$

The following example shows that Theorem 3.9 does not hold for an infinite irreducible cover.

FIGURE 3. A cover of the full shift on  $\{0, 1, 2\}$  which is not semi-open.

**Example 3.10.** Let  $\mathcal{G} = (G, \mathcal{L})$  be the cover shown in Figure 3 and  $e \in \mathcal{E}(G)$  with  $i(e) = I_1$  and  $t(e) = I_0$ . Note that  $X_{\mathcal{G}}$  is a full-shift on  $\{0, 1, 2\}$ . One way to see this is that for  $r \in \mathbb{N}$  and  $m_i \in \mathbb{N}_0$  and any  $a_j \in \{0, 1, 2\}$ , we have a path labeled  $a_1^{m_1} a_2^{m_2} \dots a_r^{m_r}$ . This implies that for any periodic point  $p^\infty \in \{0, 1, 2\}^{\mathbb{Z}}$ , we have a path  $\pi$  such that  $\mathcal{L}(\pi) = p$ . The conclusion follows since the periodic points are dense in our space.

Now suppose  $\mathcal{L}_\infty([e])$  contains a central  $2n+1$  cylinder  $[b_{-n} \dots b_n]$  with  $b_0 = 0$ . By the fact that our system is a full-shift, the point  $x = \dots 2^{n+2} b_{-n} \dots .0 \dots b_n \dots \in [b_{-n} \dots b_n]$ ; however  $x \notin \mathcal{L}_\infty([e])$ . This is because, we have  $i(b_{-n}) = I_i$ ,  $-n+1 \leq i \leq n+1$ . For instance, if  $i = n+1$ , then  $b_{-1} = b_{-2} = \dots b_{-n} = 0$  and if  $0 \leq i < n+1$ , then at least one  $b_j$  equals 2. Now let  $A_j = \{i \in \mathbb{N} : 2^i \in P(I_j)\}$  for  $j \in \mathbb{Z}$  where  $P(I_j)$  is the collection of labels of paths terminating at  $I_j$ . Then,  $A_{-1} = A_0 = \emptyset$  and  $\max A_j = j$  for  $j \in \mathbb{N}$  and  $\max A_j = -j-1$  for  $j \in \{-n : n \geq 2\}$ . So if a path  $\pi$  is labeled  $2^{n+2} b_{-n} \dots .0$ , then  $t(\pi) \neq I_0$ . This means that  $\phi$  is not semi-open.

**Corollary 3.11.** Suppose  $X$  is an irreducible sofic and let  $\phi : X \rightarrow Y$  be a factor code. Then,  $\phi$  is semi-open.

*Proof.* Let  $\mathcal{G} = (G, \mathcal{L})$  be the Fischer cover of  $X$ . Then,  $\phi \circ \mathcal{L}_\infty$  is an irreducible finite cover for  $Y$  and by Theorem 3.9, it is semi-open. Now the proof is a consequence of Lemma 3.1.  $\square$

**Theorem 3.12.** Let  $T : X \rightarrow X$  and  $S : Y \rightarrow Y$  be transitive homeomorphisms and  $\phi : X \rightarrow Y$  a factor map. Suppose there is an open set  $U_0$  such that for any open set  $V \subseteq U_0$ ,  $\text{int}(\phi(V)) \neq \emptyset$ . Then,  $\phi$  is semi-open.

*Proof.* Let  $U$  be an arbitrary open set. Since  $T$  is transitive, there is an integer  $n$  such that  $U_0 \cap T^{-n}(U)$  is a non-empty open set. By hypothesis, there is open set  $V \subseteq \phi(U_0 \cap T^{-n}(U))$  and for that

$$S^n(V) \subseteq S^n \phi(U_0 \cap T^{-n}(U)) = \phi T^n(U_0 \cap T^{-n}(U)) = \phi(T^n(U_0) \cap U) \subseteq \phi(U).$$

and since  $S$  is homeomorphism,  $S^n(V)$  is open.  $\square$

An immediate consequence of this theorem is that if  $X$  and  $Y$  are subshifts with  $T$  and  $S$  shift maps, then image of any cylinder under  $\phi$  has empty interior if and only if there is just one cylinder in  $X$  with empty interior under  $\phi$ . Example 3.10 is a good example to see this fact.

#### 4. OPEN VS SEMI-OPEN CODES

The next two theorems say that [10, Proposition 3.3] and [10, Proposition 3.4] that were stated for open codes are actually valid for semi-open codes as well. The

main ingredient in the proof is to have an open set lying in the image of any open set which is supplied by any semi-open map.

**Theorem 4.1.** *Let  $X$  be a shift of finite type,  $Y$  an irreducible sofic shift and  $\phi : X \rightarrow Y$  a finite-to-one semi-open code. Then,  $X$  is non-wandering and all components are maximal.*

Example 3.6 shows that semi-openness is required in the hypothesis of the above theorem.

**Theorem 4.2.** *Let  $\phi : X \rightarrow Y$  be a right-closing semi-open code from a shift space  $X$  to an irreducible shift of finite type  $Y$ . Then,  $X$  is a non-wandering shift of finite type.*

By Theorem 3.7, unlike open codes [10, Corollary 4.3], a right-closing semi-open extension of an irreducible strictly almost Markov (resp. non-AFT sofic) shift is not necessarily strictly almost Markov (resp. non-AFT sofic).

We have some similarities between open and semi-open codes. For instance, a coded system  $X$  is SFT if and only if there is a cover  $\mathcal{G} = (G, \mathcal{L})$  so that  $\mathcal{L}_\infty$  is open which can be deduced from [11, Proposition 3.1.6] and [10, Proposition 2.3]. A similar result for semi-open codes is Theorem 3.7. Also, SFT is invariant under open codes as synchronized systems under semi-open codes:

**Theorem 4.3.** *Suppose  $X$  is a synchronized system and  $\phi : X \rightarrow Y$  a semi-open code. Then,  $Y$  is synchronized.*

*Proof.* The Fischer cover of  $X$  will be a cover for  $Y$  whose cover map is semi-open; this fact and the conclusion is supplied by Theorem 3.7.  $\square$

Note that any coded system  $Y$  can be a factor of some synchronized systems [6, Proposition 4.1]. In particular, if  $Y$  is not synchronized, then by the above theorem, the associated code is not semi-open.

**Example 4.4.** Unlike codes between irreducible sofic shifts which are all semi-opens (Corollary 3.11), there can be a non-semi-open code between two synchronized systems. For an example, let  $Y'$  be a non-synchronized system on  $\mathcal{A}' = \{0, 1, \dots, k-1\}$  generated by a prefix code  $C'_{Y'} = \{ua', w_1, \dots\}$ ,  $a' \in \mathcal{A}'$  as in [6, Proposition 4.1]. This means if  $v \in C'_{Y'}$ , then no other element of  $C'_{Y'}$  starts with  $v$  and we know that any coded system has a prefix code [6]. Let  $\phi' : X' \rightarrow Y'$  be the one-block code between the synchronized system  $X'$  generated by  $C'_{X'} = \{uz, w_1, \dots\}$ ,  $z \notin \mathcal{A}' \cup \{k\} = \{0, 1, \dots, k\}$  and  $\phi'$  induced by the block map  $\Phi'$  defined as

$$(4.1) \quad \Phi'(x)_i = \begin{cases} x_i & x_i \in \mathcal{A}', \\ a' & x_i = z. \end{cases}$$

By Theorem 4.3,  $\phi'$  is not semi-open.

Now let  $X$  and  $Y$  be the synchronized systems generated by  $C_X = C'_{X'} \cup \{k\}$  and  $C_Y = C'_{Y'} \cup \{k\}$  respectively. Here,  $k$  is a synchronizing word for both systems. Also, define  $\phi : X \rightarrow Y$  as  $\phi'$  when restricted to  $X'$  and in other places by  $\Phi(k) = k$ . Then, since  $\phi'$  is not semi-open,  $\phi$  is not semi-open either.

Jung in [10] shows that when  $\phi$  is a finite-to-one code from a shift of finite type  $X$  to an irreducible sofic shift  $Y$ , then  $\phi$  is open if and only if it is constant-to-one. For semi-open codes we have



**Theorem 4.5.** *Let  $\phi$  be a finite-to-one code from a shift of finite type  $X$  to an irreducible sofic shift  $Y$ . If  $\phi$  is semi-open, then it will be constant-to-one a.e..*

*Proof.* By Theorem 4.1,  $X$  is non-wandering and all components are maximal. Since each component is SFT and  $\phi$  is finite-to-one, the restriction of  $\phi$  to each component is constant-to-one a.e. [11, Theorem 9.1.11]. So  $\phi$  is constant-to-one a.e. as well.  $\square$

**Remark 4.6.** Example 3.6 shows that the converse of Theorem 4.5 does not hold necessarily. However, when  $X$  is an irreducible shift of finite type and  $\phi : X \rightarrow Y$  a finite-to-one factor code, then,  $\phi$  is semi-open and constant-to-one a.e. (Corollary 3.11 and [11, Theorem 9.1.11]).

Let  $\phi : X \rightarrow Y$  be an open code between shift spaces. If  $\phi$  is bi-closing and  $Y$  irreducible, then  $\phi$  is constant-to-one [10, Corollary 2.8]. The following example shows that this result is not necessarily true for semi-open codes.

**Example 4.7.** Let  $Y$  be the even shift with its Fischer cover  $\mathcal{G} = (G, \mathcal{L})$  in Figure 2. Note that  $\mathcal{L}_\infty$  is bi-resolving and so  $Y$  is an AFT. By Theorem 3.7,  $\mathcal{L}_\infty$  is semi-open. But  $|\mathcal{L}_\infty^{-1}(y)| = 1$  for all  $y \neq 0^\infty$  and equals 2 for  $y = 0^\infty$ . So  $\mathcal{L}_\infty$  is not constant-to-one.

Let  $X, Y$  and  $Z$  be shift spaces and  $\phi_1 : X \rightarrow Z, \phi_2 : Y \rightarrow Z$  the codes. Then, the *fiber product* of  $(\phi_1, \phi_2)$  is the triple  $(\Sigma, \psi_1, \psi_2)$  where

$$\Sigma = \{(x, y) \in X \times Y : \phi_1(x) = \phi_2(y)\}$$

and  $\psi_1 : \Sigma \rightarrow X$  is defined by  $\psi_1(x, y) = x$ ; similarly for  $\psi_2$ . Let  $\phi_1$  be onto. Then, if  $\psi_1$  is open, then so is  $\phi_2$  [10, Lemma 2.4]. We do not know this result for semi-openness. However, we have the following.

**Theorem 4.8.** *Let  $(\Sigma; \phi_1, \phi_2)$  be the fiber product of  $\phi_1 : X \rightarrow Z$  and  $\phi_2 : Y \rightarrow Z$ . If  $\psi_1$  and  $\phi_1$  are semi-open and  $\psi_2$  is onto, then  $\psi_2$  and  $\phi_2$  are semi-open.*

*Proof.* If  $\psi_1$  and  $\phi_1$  are semi-open, then Lemma 3.1 implies that  $\phi_2$  is semi-open. To see that  $\psi_2$  is semi-open let  $C$  be the central  $2l + 1$  cylinder in  $\Sigma$  and let  $m$  be the coding length of  $\phi_2$ . Choose  $p \geq l$  so that  $\psi_1(C)$  contains a central  $2p + 1$  cylinder and let  $x \in [x_{-p} \cdots x_p] \subseteq \psi_1(C)$ . Also, let  $q \geq l$  so that  $\phi_1([x_{-p} \cdots x_p])$  contains a central  $2q + 1$  cylinder and let  $z \in [z_{-q} \cdots z_q] \subseteq \phi_1([x_{-p} \cdots x_p])$ . Then,  $z = \phi_1(\bar{x})$  where  $\bar{x} \in [x_{-p} \cdots x_p]$ . So there is  $\bar{y} \in Y$  such that  $(\bar{x}, \bar{y}) \in C$ .

Now we claim that  $[\bar{y}_{-q-m} \cdots \bar{y}_{q+m}] \subseteq \psi_2(C)$ . For  $y'_{[-q-m, q+m]} = \bar{y}_{[-q-m, q+m]}$ , we have  $\phi_2(y')_{[-q, q]} = \phi_2(\bar{y})_{[-q, q]}$  which means that  $\phi_2(y') \in [z_{-q} \cdots z_q]$ . So there is  $x' \in [x_{-p} \cdots x_p]$  such that  $\phi_1(x') = \phi_2(y')$  and this in turn means that  $(x', y') \in \Sigma$ . But  $y'_{[-l, l]} = \bar{y}_{[-l, l]}$  and so in fact  $(x', y') \in C$ . Thus  $\psi_2(C)$  contains  $[\bar{y}_{-q-m} \cdots \bar{y}_{q+m}]$  and the claim is established.  $\square$

**4.1. Lifting the semi-open code between synchronized systems to a code between their Fischer covers.** We usually read most of the properties of a synchronized system from its Fischer cover, so lifting the codes between systems to codes between their respective Fischer covers could be helpful.

**Lemma 4.9.** *Let  $X_i, i = 1, 2$  be a synchronized system with the Fischer cover  $\mathcal{G}_i = (G_i, \mathcal{L}_i)$ . Then, a code  $f : X_1 \rightarrow X_2$  lifts to the Fischer covers. That is, there is a code  $F : X_{G_1} \rightarrow X_{G_2}$  such that  $\mathcal{L}_{2\infty} \circ F = f \circ \mathcal{L}_{1\infty}$ .*



*Proof.* If  $t \in X_1$  is doubly transitive, then  $f(t)$  is doubly transitive in  $X_2$  and so by the fact that the degree of  $\mathcal{L}_{i\infty}$  is one [11, Proposition 9.1.6],  $F = \mathcal{L}_{2\infty}^{-1} \circ f \circ \mathcal{L}_{1\infty}$  makes sense on doubly transitive points. Pick  $x \in X_{G_1}$  and choose a sequence  $\{x_n\}_{n \in \mathbb{N}}$  of doubly transitive points converging to  $x$ . We will show that  $\lim_{n \rightarrow \infty} F(x_n)$  exists. First note that  $F(x_n)$  is a doubly transitive point for all  $n \in \mathbb{N}$ . Let  $u = \mathcal{L}_2(\pi)$  be a synchronizing word for  $X_2$  and  $w$  any word such that  $uwu \in \mathcal{B}(X_2)$ . If  $\pi$  terminates at a vertex  $I$ , then since  $\mathcal{L}_{2\infty}$  is right-resolving, there is a unique path in  $G_2$  with initial vertex  $I$  representing  $w$ . Also, any word occurs in a doubly transitive point infinitely often to the left and to the right; therefore, arbitrary central paths of  $F(x_n)$  and  $F(x_m)$  are the same for sufficiently large  $m$  and  $n$  and since we are on paths,  $\lim_{n \rightarrow \infty} F(x_n)$  exists. Set  $F(x) := \lim_{n \rightarrow \infty} F(x_n)$ .

Now we claim that  $F$  is continuous. Let  $\{z_n\}_n$  be a sequence in  $X_{G_1}$  such that  $z_n \rightarrow z$ . It is sufficient to show that  $F(z_n) \rightarrow F(z)$ .

For all  $n$ , suppose  $\{w_m^{(n)}\}_m$  is a sequence of doubly transitive points in  $X_{G_1}$  such that  $w_m^{(n)} \rightarrow z_n$ . Then, we have

$$\lim_{(m,n) \rightarrow \infty} w_m^{(n)} = z.$$

So by definition,  $F(z) = \lim_{(m,n) \rightarrow \infty} F(w_m^{(n)})$ . On the other hand, for all  $n$ ,  $F(z_n) = \lim_{m \rightarrow \infty} F(w_m^{(n)})$  and this proves the claim and we are done.  $\square$

**Theorem 4.10.** *By the hypothesis of the above lemma,*

- (1)  *$f$  is irreducible if and only if  $F$  is.*
- (2) *If  $F$  is semi-open, then  $f$  is semi-open. When  $G_1$  and  $G_2$  are locally-finite, the converse is also true.*

*Proof.* (1) is a consequence of Lemma 3.2 and Theorem 3.8.

For the proof of (2), let  $F$  be semi-open and  $U$  an open subset of  $X_1$  and set

$$g := \mathcal{L}_{2\infty} \circ F (= f \circ \mathcal{L}_{1\infty}).$$

Then, there is an open set  $V \subseteq g(\mathcal{L}_{1\infty}^{-1}(U))$ . But  $g(\mathcal{L}_{1\infty}^{-1}(U)) \subseteq f(U)$  and so  $V \subseteq f(U)$  and consequently  $f$  is semi-open.

For the converse suppose  $f$  is semi-open and let  $U$  be an open subset of  $X_{G_1}$ . Since by Theorem 3.7,  $g$  is semi-open, we may assume that there is  $V$ , an open subset of  $X_2$ , such that  $g(U) = V$ ; otherwise, replace  $U$  with  $U \cap g^{-1}(V)$ ,  $V \subseteq g(U)$ . Also, by the fact that  $X_{G_i}$  is locally compact, we may assume that  $\overline{U}$  is a compact subset of  $X_{G_1}$  and so  $X_{G_2} \setminus F(\overline{U})$  an open subset of  $X_{G_2}$ . Furthermore, we choose  $U$  so small so that  $X_{G_2} \setminus F(\overline{U}) \neq \emptyset$ . By these assumptions, we will show that  $W = F(U)$  is open in  $X_{G_2}$ . Suppose the contrary. Thus there is  $w \in W$  which is a boundary point of  $W$  and then indeed  $w \in \partial F(\overline{U})$ . Choose a sequence of doubly transitive points  $\{x_n\}$  in  $X_{G_2} \setminus F(\overline{U})$  approaching  $w$ . Since  $X_{G_2} \setminus F(\overline{U})$  is open, such a sequence exists and

$$\lim_{n \rightarrow \infty} \mathcal{L}_{2\infty}(x_n) = \mathcal{L}_{2\infty}(w) \in V.$$

But  $\mathcal{L}_{2\infty}$  is one-to-one on the set of doubly transitive points which means that  $\mathcal{L}_{2\infty}(x_n) \in X_2 \setminus V$  and this in turn implies that our open set  $V$  contains a boundary point which is absurd.  $\square$

Now let  $X$  be a non-SFT sofic shift with the Fischer cover  $\mathcal{G} = (G, \mathcal{L})$  and let  $f = \mathcal{L}_\infty$  and  $F = Id : X_G \rightarrow X_G$ . Then,  $F$  is open while  $f$  cannot be open [10, Proposition 2.3]. So Theorem 4.10 does not hold for open maps.

**4.2. On the bi-continuing codes and bi-closing codes.** A code  $\phi : X \rightarrow Y$  is called *right-continuing almost everywhere (a.e.)* if whenever  $x$  is left transitive in  $X$  and  $\phi(x)$  is left asymptotic to a point  $y \in Y$ , then there exists  $\bar{x} \in X$  such that  $\bar{x}$  is left asymptotic to  $x$  and  $\phi(\bar{x}) = y$ . Similarly we have the notions called left-continuing a.e. and bi-continuing a.e.

If  $\phi : X \rightarrow Y$  is open with a uniform lifting length, then it is bi-continuing (everywhere) with a bi-retract [9, Lemma 2.1]. The following shows that a semi-open code with a uniform lifting length is not necessarily bi-continuing a.e. with a bi-retract.

**Example 4.11.** This example was constructed in [9] to show that a continuing code may not have a retract; we use it for our prementioned purpose.

Let  $X$  be a shift space on the alphabet  $\{1, \bar{1}, 2, 3\}$  defined by forbidding  $\{\bar{1}2^n 3 : n \geq 0\}$ , and let  $Y$  be the full 3-shift  $\{1, 2, 3\}^{\mathbb{Z}}$ . Define  $\phi = \Phi_\infty : X \rightarrow Y$  by letting  $\Phi(\bar{1}) = 1$  and  $\Phi(a) = a$  for all  $a \neq \bar{1}$ . The map  $\phi$  is bicontinuing and it has (left continuing) retract [9]. Now for  $n \in \mathbb{N}$ , consider a left transitive point  $x = \cdots \bar{1}2^n.22^\infty$  and pick  $y = \cdots 12^n.23^\infty \in Y$  so that  $\phi(x)$  is left asymptotic to  $y$ . One can readily see that  $\phi$  does not have right continuing a.e. retract.

We show that  $\phi$  is semi-open with a uniform lifting length. Let  $[a_{-n} \cdots a_n]$  be a central  $2n + 1$  cylinder in  $X$  and  $\Phi(a_i) = b_i$ ,  $-n \leq i \leq n$ . Thus if  $a_i \neq \bar{1}$ , then  $\phi([a_{-n} \cdots a_n]) = [b_{-n} \cdots b_n]$ ; otherwise,  $[1b_{-n} \cdots b_n 1] \subseteq [b_{-n} \cdots b_n 1] \subseteq \phi([a_{-n} \cdots a_n])$ . So  $\phi$  is semi-open with a uniform lifting length as required.

If  $X$  is an irreducible sofic shift, then by [11, Corollary 4.4.9],

$$(4.2) \quad h(Y) < h(X), \text{ } Y \text{ is a proper subsystem of } X.$$

This condition is sufficient to have double transitivity a totally invariant property for finite-to-one factor codes. That is,

**Lemma 4.12.** *Suppose  $X$  is compact and  $\phi : X \rightarrow Y$  a finite-to-one factor code. If either  $X$  satisfies (4.2) or  $\phi$  is irreducible, then  $x \in X$  is doubly transitive if and only if  $\phi(x)$  is.*

*Proof.* When  $X$  satisfies (4.2), it is a consequence of [1, Theorem 3.4]. So suppose  $\phi$  is irreducible. If  $\phi(x)$  is doubly transitive, but  $x$  is not, then the proof of [11, Lemma 9.1.13] implies that there will be a proper subshift  $Z$  of  $X$  with  $\phi(Z) = Y$ ; and this contradicts the irreducibility of  $\phi$ .  $\square$

**Theorem 4.13.** *Suppose  $X$  and  $\phi$  satisfy the hypothesis of Lemma 4.12 and  $Y$  a SFT. If  $\phi$  is bi-continuing a.e. with a bi-retract, then it will be open with a uniform lifting length.*

*Proof.* Suppose that  $\phi$  is bi-continuing a.e. with a bi-retract  $n \in \mathbb{N}$ . We may assume that our SFT system  $Y$  is an edge shift and  $\phi$  is a 1-block code. Let  $l \geq 0$  and  $u \in \mathcal{B}_{2l+1}(X)$ . Choose  $x \in [u]$  to be a left transitive point and pick a doubly transitive point  $y \in Y$  with  $y_{[-l-n, l+n]} = \phi(x)_{[-l-n, l+n]}$ . The point  $\hat{y}$  given by

$$\hat{y}_i = \begin{cases} \phi(x)_i & i \leq 0, \\ y_i & i > 0. \end{cases}$$

is a doubly transitive point in  $Y$ . Since  $n$  is a right continuing a.e. retract and  $\phi(x)_{(-\infty, l+n]} = \hat{y}_{(-\infty, l+n]}$ , there is  $\bar{x} \in X$  such that  $\bar{x}_{(-\infty, l]} = x_{(-\infty, l]}$  and  $\phi(\bar{x}) = \hat{y}$ . By Lemma 4.12,  $\bar{x}$  is a doubly transitive point and we have  $\phi(\bar{x})_{[-l-n, \infty)} = \hat{y}_{[-l-n, \infty)} = y_{[-l-n, \infty)}$ . By the fact that  $n$  is also a left continuing a.e. retract, there is  $\bar{\bar{x}} \in X$  such that  $\bar{\bar{x}}_{[-l, \infty)} = \bar{x}_{[-l, \infty)}$  and  $\phi(\bar{\bar{x}}) = y$ . Note that  $\bar{\bar{x}}_{[-l, l]} = \bar{x}_{[-l, l]} = x_{[-l, l]}$  which means that  $\bar{\bar{x}}$  is in  $[u]$ .

Now choose an arbitrary  $y' \in [\phi(x)_{[-l-n, l+n]}]$  and pick a sequence of doubly transitive points  $\{y^m\}_m$  in  $[\phi(x)_{[-l-n, l+n]}]$  such that  $y^m \rightarrow y'$  and let  $\bar{\bar{x}}^m$  be a point in  $[u]$  with  $\phi(\bar{\bar{x}}^m) = y^m$ . Thus for all  $m$ ,  $y^m \in \phi([u])$ . But  $\phi([u])$  is closed and so  $y' \in \phi([u])$  which implies that  $[\phi(x)_{[-l-n, l+n]}] \subseteq \phi([u])$ . This shows that  $\phi$  is semi-open.

It remains to prove that  $\phi$  is open. Let again  $u = u_{-l} \cdots u_l$  and let  $x' \in [u]$  and choose a doubly transitive point  $x$  such that  $x_{[-l-n, l+n]} = x'_{[-l-n, l+n]}$ . Since  $\phi$  is a 1-block code,  $\phi(x)_{[-l-n, l+n]} = \phi(x')_{[-l-n, l+n]}$ . Also since  $Y$  is an edge shift,  $[\phi(x')_{[-l-n, l+n]}] = [\phi(x)_{[-l-n, l+n]}]$ . By above,  $[\phi(x)_{[-l-n, l+n]}] \subseteq \phi([u])$ . But  $x'$  was arbitrary and so  $\phi$  is open with uniform lifting length.  $\square$

By Theorem 4.13 and [9, Lemma 2.1], we have the following:

**Corollary 4.14.** *Let  $X$  and  $\phi$  satisfy the hypothesis of Lemma 4.12 and  $Y$  a SFT. If  $\phi$  is bi-continuing a.e. with a bi-retract, then it will be bi-continuing (everywhere) with a bi-retract.*

There is another situation where a bi-continuing a.e. code implies bi-continuing. In fact, Ballier interested in sofics in [4], states and proves the next theorem for when  $X$  is an irreducible sofic and  $k = 0$ . His proof exploits only the irreducibility of  $X$  which is provided here. Hence we have

**Theorem 4.15.** *Suppose  $X$  is an irreducible shift space and  $Y$  an irreducible SFT. A right-continuing a.e. factor map  $\phi : X \rightarrow Y$  with retract  $k$  is right-continuing (everywhere) with retract  $k$ .*

The next theorem gives a sufficient condition for a factor code being bi-continuing a.e..

**Theorem 4.16.** *Let  $X$  be synchronized and  $\phi : X \rightarrow Y$  a semi-open code. Then,  $\phi$  is bi-continuing a.e..*

*Proof.* Let  $x = \cdots x_{-1}x_0x_1 \cdots \in X$  be a left transitive point and  $y \in Y$  such that  $\phi(x)_{(-\infty, 0]} = y_{(-\infty, 0]}$ . Since  $x$  is left transitive, any word in  $X$  happens infinitely many often on the left of  $x$ . So without loss of generality, assume that  $x_0$  is the synchronizing word.

By semi-openness, there is a cylinder  ${}_l[u]$  contained in  $\phi({}_0[x_0])$  where  ${}_l[u]$  denotes the set  $\{y \in Y : y_{[l, l+|u|-1]} = u\}$ . Since  $x$  is left transitive, there exist infinitely many  $k > 0$  such that  $\sigma^{-k}(\phi(x)) \in {}_l[u]$ , or equivalently,  $\phi(x) \in \sigma^k({}_l[u]) = {}_{l-k}[u]$ . Choose a sufficiently large  $k$  so that  $l - k + |u| - 1 < 0$ .

Now since  $\phi(x)_{(-\infty, 0]} = y_{(-\infty, 0]}$ ,  $y \in {}_{l-k}[u]$  or in fact  $y \in {}_{l-k}[v]$  where

$$v = y_{l-k} \cdots y_{-1}y_0 = u_{l-k} \cdots u_{l-k+|u|-1}y_{l-k+|u|} \cdots y_{-1}y_0.$$

We have  ${}_{l-k}[v] \subseteq {}_{l-k}[u] \subseteq \phi({}_{-k}[x_0])$ . Set  $W = \phi^{-1}({}_{l-k}[v]) \cap {}_0[x_0]$  and note that  $W$  is an open set containing  $x$  and  $y \in \phi(W)$ . Therefore, there must be  $z \in W$  such

that  $\phi(z) = y$  and define

$$\bar{x}_i = \begin{cases} x_i & i \leq 0, \\ z_i & i \geq 0. \end{cases}$$

Since  $x_0$  is a synchronizing word,  $\bar{x} \in X$  and so we have a  $\bar{x}$  which is left-asymptotic to  $x$  and  $\phi(\bar{x}) = y$ . This means  $\phi$  is right-continuing a.e.; and similarly, it is left-continuing a.e..  $\square$

#### REFERENCES

1. D. Ahmadi Dastjerdi and S. Jangjoo, On synchronized non-sofic subshifts, arXiv:1303.6570.
2. E. Akin, E. Glasner, W. Huang, S. Shao and X. Ye, Sufficient conditions under which a transitive system is chaotic, *Ergo. Th. & Dynam. Sys.* **30**, No. 5, 1277-1310 (2010).
3. J. Auslander, *Minimal flows and their extensions*, North-Holland Math. Stud. 153, North-Holland, 1988.
4. Ballier, A.: Limit sets of stable cellular automata, *Ergo. Th. & Dynam. Sys.*, DOI: 10.1017/etds.2013.72, (2013).
5. A. Bella, A. Baszczyk, A. Szymaski: On absolute retracts of  $\omega^*$ . *Fund. Math.* **145** (1994), 1-13.
6. F. Blanchard and G. Hansel, Systèmes codés, *Theoretical Computer Science.* **44** 14-49, 1986.
7. M. Boyle and S. Tuncel, Infinite-to-one codes and Markov measures, *Trans. Amer. Math. Soc.* **285** (1984), 657-684.
8. D. Fiebig and U. Fiebig, Covers for coded systems, *Contemporary Mathematics*, **135**, 1992, 139-179.
9. U. Jung, On the existence of open and bi-continuing codes, *Trans. Amer. Math. Soc.* **363** (2011), 13991417.
10. U. Jung, Open maps between shift spaces, *Ergo. Th. & Dynam. Sys.* **29** (2009), 12571272.
11. D. Lind and B. Marcus, *An introduction to symbolic dynamics and coding*, Cambridge Univ. Press, 1995.
12. T. K. Subrahmonian Moothathu, Studies in topological dynamics with emphasis on cellular automata, PhD thesis, Department of Mathematics and Statistics, School of MCIS, University of Hyderabad, (2006).
13. P. Walters, *An introduction to ergodic theory*, Springer-Verlag, 1982.
14. G. T. Whyburn, *Analytic topology*, **28**, AMS Coll. Publications, Providence, RI, 1942.

FACULTY OF MATHEMATICS , UNIVERSITY OF GUILAN

E-mail address: dahmadi1387@gmail.com, sjangjoo90@gmail.com